The De Giorgi–Nash–Moser Regularity Theory

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Introduction

These lessons are devoted to the study of the Hölder regularity of solutions to linear elliptic equations in divergence form with measurable and bounded coefficients (therefore possibly discontinuous). Ennio De Giorgi first proved the result in 1957 (see **[1]**) to solve the Hilbert's 19th problem which consisted in showing that local minimizers of an energy functional

$$\mathcal{F}(w) = \int_\Omega F(Dw)\,dx$$

are regular provided F is regular. Here Ω is an open set of $\mathbb{R}^N$ and $F : \mathbb{R}^N \to \mathbb{R}$ is such that, for every $p \in \mathbb{R}^N$,

$$\lambda|\xi|^2 \le D^2F(p)\xi\cdot\xi \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^N.$$

A local minimizer means that $\mathcal{F}(w) \le \mathcal{F}(w+\varphi)$, for any φ compactly supported in Ω. It is standard to show that such a minimizer w is a local weak solution to the Euler–Lagrange equation

$$\operatorname{div}(DF(Dw)) = 0 \quad \text{in } \Omega, \tag{1}$$

that is

$$\int_\Omega DF(Dw)\cdot D\varphi\,dx = 0,$$

for every function $\varphi \in C_c^\infty(\Omega)$, w being in the energy space $W^{1,2}_{\mathrm{loc}}(\Omega)$.

Note that (1) can be formally written in non divergence form as

$$D^2F(Dw)\cdot D^2w = 0.$$

From the standard Calderon–Zygmund theory (which was known at the time), if Dw is $C^{0,\alpha}$, we can see this equation as a linear equation on w with elliptic $C^{0,\alpha}$ coefficients, by freezing the dependance on w in $D^2F(Dw)$. This provides $C^{1,\alpha}$ regularity on Dw and so $C^{2,\alpha}$ regularity on w. Bootstrapping the argument you obtain finally C^∞ regularity on w.

However, at this point, we have only $Dw \in L^2$ (since you can prove that $w \in W^{2,2}_{\mathrm{loc}}(\Omega)$) and this theory does not work for weak solutions.

In this context De Giorgi proved the following theorem.

THEOREM 0.1. *Let Ω be a bounded open set of $\mathbb{R}^N$ and $\lambda, \Lambda > 0$. Consider $A(x)$ a measurable $N\times N$–matrix valued function defined on Ω such that*

$$\lambda|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda|\xi|^2 \text{ a.e in } \Omega, \quad \forall \xi \in \mathbb{R}^N.$$

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a weak solution to*

$$-\mathrm{div}(A(x)Du) = 0 \quad \textit{in } \Omega.$$

Then $u \in C^{0,\alpha}(\Omega')$ *for any* $\Omega' \subset\subset \Omega$*, with*

$$\|u\|_{C^{0,\alpha}(\Omega')} \le C\|u\|_{L^2(\Omega)},$$

where the constant α *depends only on* λ, Λ, N*, while the constant* C *depends also on* Ω *and* Ω'*.*

Thus, considering, for every $1 \le i \le N$, the derivative with respect to x_i of (1) and denoting $u = \partial_i w$ it holds that u solves in a weak sense the equation

$$\mathrm{div}(D^2F(Dw)Du) = 0.$$

By applying Theorem 0.1 this gives that $Dw \in C^{0,\alpha}$. Then, with a bootstrap argument, the Calderon–Zygmund result gives the solution to the Hilbert problem.

The same result was proved independently by John Nash in 1958 (for the parabolic case). Later Jürgen Moser gave an alternative proof (see [**2**]), and the resulting theory is often called the De Giorgi–Nash–Moser theory.

In the first part of this course we present the original method introduced by De Giorgi to prove Hölder regularity of solutions to elliptic problems with rough coefficients, while, in the second part, we present Moser's technique and Harnack's inequality that as a corollary yield the De Giorgi's theorem.

References

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2. J. Moser, A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.*, **13** (1960), 457-468.

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